Existence of Geometric Ergodic Periodic Measures of Stochastic Differential Equations

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I. Random periodic processes and periodic measures

Consider measurable random dynamical system on the measurable space $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ over a metric dynamical system $(\Omega, \mathcal{F}, P, (\theta_s)_{s\in R})$:

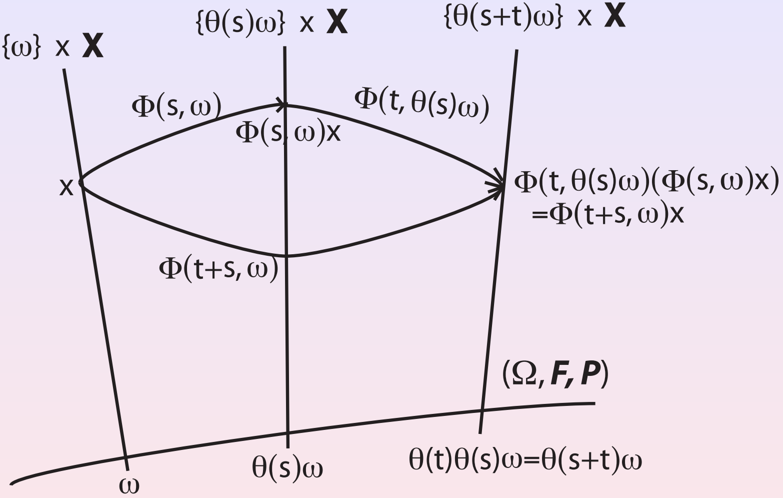
$$\Phi : R^+ \times \Omega \times \mathbb{X} \to \mathbb{X}, \quad (t, \omega, x) \mapsto \Phi(t, \omega, x),$$

with the following properties:
(i) Measurability: Φ is $(\mathcal{B}(R^+) \otimes \mathcal{F} \otimes \mathcal{B}(\mathbb{X}), \mathcal{B}(\mathbb{X}))$-measurable.
(ii) Cocycle property: for almost all $\omega \in \Omega$

$$\Phi(0, \omega) = id_{\mathbb{X}}$$

$$\Phi(t + s, \omega) = \Phi(t, \theta_s\omega) \circ \Phi(s, \omega) \ \ \textit{for all } s, t \in R^+.$$

Applicable to: SDEs, SPDEs, Markov chains, random mappings.


{ω} x X
{θ(s)ω} x X
{θ(s+t)ω} x X
Φ(s, ω)
Φ(t, θ(s)ω)
Φ(s, ω)x
x
Φ(t, θ(s)ω)(Φ(s, ω)x)
=Φ(t+s, ω)x
Φ(t+s, ω)
(Ω, F, P)
ω
θ(s)ω
θ(t)θ(s)ω=θ(s+t)ω

Definition 1

(Z. and Zheng JDE (2009), Feng, Z. and Zhou JDE (2011), Feng and Z. JFA (2012)) Let $\Phi : \mathbb{R}^+ \times \Omega \times \mathbb{X} \to \mathbb{X}$ be a random dynamical system.

A random periodic path of period τ is an $\mathcal{F}$-measurable function $Y : R \times \Omega \to \mathbb{X}$ such that for a.e. $\omega \in \Omega$,

$$\Phi(t, \theta_s\omega)Y(s, \omega) = Y(t + s, \omega), \quad Y(\tau + s, \omega) = Y(s, \theta_\tau\omega), \tag{1}$$

for all $t \in \mathbb{R}^+, s \in \mathbb{R}$.

The concept of RPP led to progress in:
–stochastic bifurcations: Wang (Nonlinear analysis 2014)
–random attractors: Bates-Lu-Wang (Physica D 2014)
–stochastic resonance: Cherubini-Lamb-Rasmussen-Sato (Nonlinearity 2017)
–stochastic horseshoe: Lian-Huang (Dec 2016)
–climate dynamics: (Chekroun, Simonnet and Ghil, Physica D (2011))
–Periodic measures and ergodic theory:
Feng-Wu-Z. (JFA 2016); Feng-Liu-Z. (ZAMP 2017); Feng-Z. (2018); Feng-Qu-Z. (2018); Feng-Z. -Zhong (2019).
In pipelines
Feng-Liu-Z. ; Feng-Z. -Zhong; Feng-Qu-Z. ; Feng-YJ Liu-Z. .

Two kinds of problems:

(1) Cocycles:

• Assume a deterministic system has a periodic path, what happens when it is affected by noise?
• Random mappings.

(2) Time-periodic semi-flows (SDEs or SPDEs with time dependent coefficients being periodic in time).

Periodic semi-flows can be lifted to a cocycle on a cylinder.

Example 2

Consider the following stochastic differential equation on $\mathbb{R}^2$

$$\begin{cases} dx = & [-y + x(1 - x^2 - y^2)]dt + xdW_1(t), \\ dv = & [x + y(1 - x^2 - y^2)]dt + ydW_2(t). \end{cases} \tag{2}$$

Periodic solution of the deterministic system: $x = \cos t, y = \sin t$.
Define $\theta : R \times \Omega \to \Omega$ as

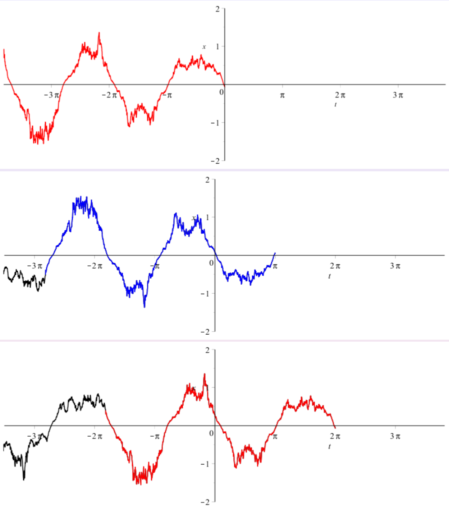
$$(\theta_t \omega)(s) = W(t + s) - W(t).$$

Proposition 3

(Feng and Z. (2018)) Equation (2) has a random periodic solution $(x^*(t), y^*(t)) \neq (0, 0)$ *with a positive minimum period satisfying*

$$(x^*(t + \pi, \omega), y^*(t + \pi, \omega)) = -(x^*(t, \theta_\pi \omega), y^*(t, \theta_\pi \omega)), \tag{3}$$

$$(x^*(t + 2\pi, \omega), y^*(t + 2\pi, \omega)) = (x^*(t, \theta_{2\pi} \omega), y^*(t, \theta_{2\pi} \omega)). \tag{4}$$



$$\text{Fig}: \quad x^*(t+2\pi,\omega) = x^*(t,\theta_{2\pi}\omega), \quad x^*(t+\pi,\omega) = -x^*(t,\theta_\pi\omega)$$

Markovian RDS set up

Consider a Markovian cocycle random dynamical system Φ on a separable Banach space $\mathbb{X}$.

Recall for any $\Gamma \in \mathcal{B}$

$$P(t,x,\Gamma) = P\{\omega : \Phi(t,\omega)x \in \Gamma\},\ t \in \mathbb{R}^+,$$

and $P^*(t) : \mathcal{P}(\mathbb{X}) \to \mathcal{P}(\mathbb{X})$ by: for any measure ρ on $\mathcal{B}$,

$$(P_t^*\rho)(\Gamma) = \int_{\mathbb{X}} P(t,x,\Gamma)\rho(dx),\ t \in \mathbb{R}^+.$$

Definition 4

(Feng-Z. (2014)) A measure function $\{\rho_s\}_{s\in R}$ *in* $\mathcal{P}(\mathbb{X})$ *is a* *periodic measure on* $(\mathbb{X},\mathcal{B})$ *if*

$$\rho_{\tau+s}=\rho_s,\quad P_t^*\rho_s=\rho_{t+s},\ t\in R^+ \tag{5}$$

Remark 5

(1) Note for each fixed s, ρ_s *is invariant measure of* $\{P(k\tau)\}_{k\in\mathbb{N}}$.
(2) $\bar{\rho}=\frac{1}{\tau}\int_{[0,\tau)}\rho_s ds$ *is an invariant measure of* $\{P(t)\}_{t\in\mathbb{R}^+}$.

Theorem 6

(Feng and Z. (2014))
Random periodic paths " <=> " periodic measures.

Skew product set up

$$(\mu_s)_\omega = \delta_{Y(s,\theta_{-s}\omega)}$$

gives a periodic measure $(\mu_s)_{s\in\mathbb{R}}$ w.r.t. skew product $(\bar{\Theta}_t)_{t\geq 0}$ on $\Omega \times \mathbb{X}$.

Conversely, given a periodic measure, one can enlarge the probability space and construct random periodic paths.

Markovian RDS set up

The law of the random periodic paths

$$\rho_s(\Gamma) = P\{\omega : Y(s,\omega) \in \Gamma\} = \mathbb{E}(\mu_s)_\cdot,$$

is a periodic measure.

Conversely, the periodic measure $\rho_\cdot$ on $\mathbb{X}$ => a periodic measure w.r.t. $\bar{\Theta}_\cdot$ on $\Omega \times \mathbb{X}$.

III. Semiflows: Existence of periodic measures

Consider the time dependent stochastic differential equations on R^d

$$dx(t) = b(t, x(t))dt + \sigma(t, x(t))dW(t), \ t \geq s, \ x(s) = x. \tag{6}$$

Under suitable conditions (existence and uniqueness of solutions), it generates stochastic semi-flow:

$$\Phi(t, s) = \Phi(t, r) \circ \Phi(r, s), \text{ for all } s \leq r \leq t, \ a.s.$$

Assume b, σ are periodic in t with the same period τ, then

$$\Phi(t + \tau, s + \tau, \omega) = \Phi(t, s, \theta_\tau \omega) \ \ a.e.\omega.$$

Its transition probability $P(t, s, x, \Gamma) = P\{\omega : \Phi(t, s, \omega)x \in \Gamma)$ satisfies

$$P(t + \tau, s + \tau, x, \Gamma) = P(t, s, x, \Gamma). \tag{7}$$

A τ-periodic measure is a measure valued function $\rho : \mathbb{R} \to \mathcal{P}(\mathbb{X})$ such that for any $s \in \mathbb{R}$, $t \geq s$,

$$\int_{\mathbb{X}} P(t, s, x, \Gamma)\rho_s(dx) = \rho_t(\Gamma), \quad \rho_{s+\tau} = \rho_s. \tag{8}$$

Note

$$P(s + \tau, s, x, \cdot)$$

is like the transition probability of an one-step Markov chain.

Invariant measures of Markov chains: Meyn and Tweedie (1992).

Let P be a one-step time-homogeneous Markov transition kernel. We say that P satisfies the “local Doeblin” condition if there exists a non-empty measurable set $K \in \mathcal{B}$, constant $\eta \in (0, 1]$ and a probability measure φ with $\phi(K) = 1$ such that

$$P(x, \cdot) \geq \eta\varphi(\cdot), \quad x \in K. \tag{9}$$

Theorem 7

Assume $P(s_* + \tau, s_*)$ satisfies the local Doeblin condition and there exist $s_* \in \mathbb{T}$, a norm-like function $U_{s_*} : \mathbb{X} \to \mathbb{R}^+$ such that

$$P(s_* + \tau, s_*)U_{s_*} \leq \alpha U_{s_*} + \beta, \quad \text{on } \mathbb{X},$$

where $\alpha \in (0,1)$ and $\beta > 0$. Then there exist a unique geometric periodic measure ρ. and a norm-like function $V : \mathbb{T} \times \mathbb{X} \to \mathbb{R}^+$, constants $R_s < \infty$ and $r_s \in (0,1)$ such that for any $s \leq t$, $x \in \mathbb{X}$, we have

$$||P(t, s - n\tau, x, \cdot) - \rho_t||_{TV} \leq R_s(V(s,x) + 1)r_s^n.$$

Norm-like: $U_{s^*}(x) \to \infty$ when $||x|| \to \infty$.

Assume b is locally Lipschitz in x and

$$(x-y, b(t,x)-b(t,y)) \le C - \lambda \|x-y\|^2. \tag{10}$$

σ, a $d \times d$ matrix, is of linear growth, locally Lipschitz and uniformly elliptic,

Theorem 8

(Feng-Z. -Zhong (2019)) SDE (6) has a unique periodic measure (which is geometric ergodic)

$$\|P(s,t,x,\cdot) - \rho_t\|_{TV} \sim \mathrm{e}^{-\delta t}, \quad \text{as } t \to \infty. \tag{11}$$

Similar results hold for

Gradient systems
Langevin equations

Denote

$$L_s = \frac{1}{2}\sum_{i,j=1}^{d}(\sigma(s,x)\sigma^*(s,x))_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s,x)\frac{\partial}{\partial x_i}.$$

Consider the Fokker-Planck operator

$$L_t^* q = -\sum_{i=1}^{d}\partial_{x_i}(b_i(t,y)q) + \frac{1}{2}\sum_{i,j=1}^{d}\partial^2_{y_i y_j}\left(\left(\sigma\sigma^T(t,y)\right)_{ij} q\right). \tag{12}$$

As σ is non-degenerate, the Markov transition kernel has density $p(t,s,x,y)$ which satisfies the Fokker-Planck equation

$$\partial_t p(t,s,x,y) = L_t^* p(t,s,x,y).$$

Note Fubini's theorem yields

$$\rho_t(\Gamma) = \int_{\mathbb{R}^d} P(t, s, x, \Gamma)\rho_s(dx) = \int_\Gamma \left(\int_{\mathbb{R}^d} p(t, s, x, y)\rho_s(dx) \right) dy. \quad (13)$$

If ρ is a periodic measure, then specifically

$$\rho_t(\Gamma) = \int_\Gamma \left(\int_{\mathbb{R}^d} p(t + \tau, t, x, y)\rho_t(dx) \right) dy,$$

that is to say $q(t, y) = \int_{\mathbb{R}^d} p(t + \tau, t, x, y)\rho_t(dx)$ is the density of ρ_t and

$$q(t, y) = \int_{\mathbb{R}^d} p(t, s, x, y)q(s, x)dx. \quad (14)$$

(Feng-Z.-Zhong (2019))

Theorem 9

(Feng-Z.-Zhong (2019)) Then ρ is a τ-periodic measure if and only if q satisfies the Fokker-Planck equation

$$\partial_t q = L_t^* q, \quad q(0,\cdot) = q(\tau,\cdot). \tag{15}$$

IV. Applications: stochastic resonance

Through studying deep-sea sediment cores, it was noted that peaks observed in the power spectrum of paleoclimatic variations in the last 700,000 years occur at a periodicity of around 10^5 years.

• The major peaks represent dramatic climate change with a temperature change of 10K in Kelvin scale.

• Except for the dramatic changes, temperature seems to oscillate around fixed values.

This phenomenon was suggested to be related to variations in the earth's orbital parameter which also has a similar periodic pattern of changes

Milankovitch (1930);

Hays, Imbrie and Shackleton (1976),

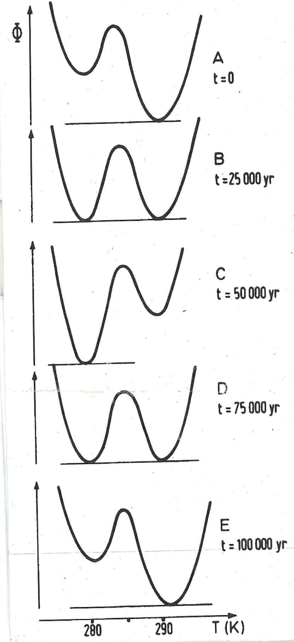
Benzi, Parisi, Sutera and Vulpiani (1982):

pointed out that such studies were able to reproduce smaller peaks, but failed to explain the 10^5-year cycle major peak.

The stochastic overdamped Duffing Oscillator was proposed:

$$dx_t = \left[-x_t^3 + x_t + A\cos(\omega t)\right] dt + \sigma dW_t, \tag{16}$$

where $A, \omega \in \mathbb{R}$ and $\sigma \neq 0$.



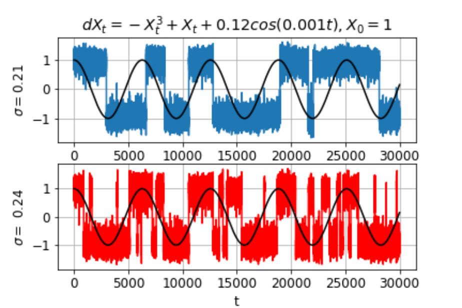
Φ
A
t = 0
B
t = 25 000 yr
C
t = 50 000 yr
D
t = 75 000 yr
E
t = 100 000 yr
280
290
T (K)

But a mathematical theory is needed.

Large deviation theory for small noise.

Noise may not be small to enable stochastic resonance–our theory provides new insight.

The uniqueness is significant in explaining the transition between the two wells as otherwise there should be two periodic measures instead of one.


$dX_t = -X_t^3 + X_t + 0.12cos(0.001t), X_0 = 1$
σ= 0.21
σ= 0.24
1
0
−1
0
5000
10000
15000
20000
25000
30000
t

Exit problem (including stochastic resonance)

Consider a smooth open domain $D \subset \mathbb{R}^d$ and define

$$\eta(s,x) = \inf_{t \geq s}\{x(t,s,x) \in D\} - s,$$

where $x \in$D, otherwise $\eta(s,x) = 0$.

Theorem 10

(Feng-Z.-Zhong (WIP))
(i) The expected exit duration $\bar{\eta}(s,x) = \mathbb{E}\eta(s,x)$ *satisfies the following PDEs on* $[0,\tau] \times D$*:*

$$\begin{cases} \partial_s u(s,x) + L_s u(s,x) = -1, \\ u = 0, \qquad \text{on } [0,\tau] \times \partial D, \\ u(0,x) = u(\tau,x). \end{cases} \tag{17}$$

(ii) Under smoothness and non-degenerate assumptions, for fixed $p > d$*, there exists a solution* u *such that* $u(0) \in L^p(D)$*.*

V. Ergodicity under random periodicity

Case of Markovian random dynamical systems

Well-known results

ρ is weakly mixing

$<=>$

there exists $I \subset [0, \infty)$ of relative measure 1 such that

$$\lim_{t \to \infty, t \in I} P(t, x, -) \to \rho$$

$<=>$

if $P(t)\phi = e^{i\lambda t}\phi$, λ is a real number, then $\lambda = 0$ and ϕ is a constant.

$<=>$

its infinitesimal generator has simple eigenvalue 0 ONLY on the complex axis

(Koopman-von Neumann)

[stationary regime/stationary processes]

More well-known results and new observation

ρ is ergodic.

<=>

a set $\Gamma \in \mathcal{B}(\mathbb{X})$ satisfies for all $t > 0$, $P_t I_\Gamma = I_\Gamma,\ \rho - a.e.$ then either $\rho(\Gamma) = 0$ or $\rho(\Gamma) = 1$.

<=>

if $P(t)\phi = \phi$, then ϕ is a constant.

<=>

its infinitesimal generator has simple eigenvalue 0

<=>

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T P(s,x,\Gamma)ds \to \rho(\Gamma),\ \text{in } L^2(\mathbb{X},\rho(dx)).$$

(Birkhoff's ergodic theorem)

[stationary & random periodic processes]

($\bar{\rho}$ as invariant measure is ergodic)

Theorem 11

(Feng-Z. (2018)) If the τ*-periodic measure* $\{\rho_s\}_{s\in\mathbb{R}}$ *has minimum period* τ *and* ρ_s *is mixing with respect to* $P(k\tau)$, $k \in \mathbb{N}$ *for each* s*, then*

• *the infinitesimal generator* $\mathcal{L}$ *has simple eigenvalues* $\{\frac{2lm\pi}{\tau} i\}_{m\in\mathbb{Z}}$*, for some* $l \in \mathbb{N}$*, and no other eigenvalues on the imaginary axis.*

Conversely, if

• *the infinitesimal generator* $\mathcal{L}$ *has simple eigenvalues* $\{\frac{2m\pi}{\tau} i\}_{m\in\mathbb{Z}}$ *and no other eigenvalues on the imaginary axis,*

then the periodic measure $\{\rho_s\}_{s\in\mathbb{R}}$ *has minimum period* τ *and* ρ_s *is ergodic with respect to* $P(k\tau)$, $k \in \mathbb{N}$ *for each* s *thus* $\bar{\rho}$ *is ergodic with respect* $P(t)$, $t \geq 0$.

Semi-flows: Lifiting

Lemma 12

(Feng-Z. (2014)) We lift the τ-periodic stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \to \mathbb{X}$ to a random dynamical system on a cylinder $\tilde{\mathbb{X}} := [0, \tau) \times \mathbb{X}$ by the following:

$$\tilde{\Phi}(t, \omega)(s, x) = (t + s \ mod \ \tau, u(t + s, s, \theta(-s)\omega)x). \tag{18}$$

Then $\tilde{\Phi} : \mathbb{R}^+ \times \Omega \times \tilde{\mathbb{X}} \to \tilde{\mathbb{X}}$ is a cocycle on $\tilde{\mathbb{X}}$ over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(s))_{s \in \mathbb{R}})$.
Moreover, assume $Y : \mathbb{R} \times \Omega \to \mathbb{X}$ is a random periodic solution of the semi-flow u with period $\tau > 0$. Then $\tilde{Y} : \mathbb{R} \times \Omega \to \tilde{\mathbb{X}}$ defined by

$$\tilde{Y}(s, \omega) := (s \ mod \ \tau, Y(s, \omega)), \tag{19}$$

is a random periodic solution of the cocycle $\tilde{\Phi}$ on $\tilde{\mathbb{X}}$.

Lifting corresponding measures:

Note

$$\begin{aligned}\tilde{P}(t,(s,x),C\times\Gamma) &= \delta_{(t+s \bmod \tau)}(C)P(t+s,s,x,\Gamma),\\ \tilde{\rho}_s(C\times\Gamma) &= \delta_{(s \bmod \tau)}(C)\rho_s(\Gamma).\end{aligned}$$

Its infinitesimal generator is

$$\mathcal{L} = L_s + \frac{\partial}{\partial s}.$$

Then

$$\begin{aligned}\tilde{P}(t)^*\tilde{\rho}_s &= \tilde{\rho}_{t+s},\\ \tilde{\rho}_{s+\tau} &= \tilde{\rho}_s.\end{aligned}$$

Moreover

$$\bar{\tilde{\rho}} = \frac{1}{\tau}\int_0^\tau \tilde{\rho}_s ds \tag{20}$$

is the invariant measure of $\tilde{P}(t),\ t \geq 0$, so its ergodic theory can be established.

Theorem 13

(Feng-Zhao (2018)) Assume for each fixed $s \in \mathbb{R}$*, the second order differential operator* L_s *on* $D(L_s)$ *is a self-adjoint operator with a simple eigenvalue* 0*.*

Then

(1) the operator $\tilde{\mathcal{L}}$ *on* $D(\tilde{\mathcal{L}})$ *has simple eigenvalues* $\lambda_m = \frac{2m\pi}{\tau} i, m \in \mathbb{Z}$*, and no other eigenvalues on the imaginary axis.*

(2) for each $s \in \mathbb{R}$*,* $\tilde{\rho}_s$ *as an invariant measure of* $\{\tilde{P}(k\tau)\}_{k\in\mathbb{N}}$ *is ergodic and thus* $\bar{\tilde{\rho}}$ *is ergodic invariant measure with respect to* $\{\tilde{P}(t)\}_{t\in\mathbb{R}^+}$*.*

THANK YOU!